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Canonical solution of the state labelling problem for SU(n) ⊃ SO(n) and Littlewood's branching rule: I. General formulation

J Deenen and C Quesne†

Service de Physique Théorique et Mathématique CP 229, Université Libre de Bruxelles, Bd du Triomphe, B 1050 Brussels, Belgium

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Abstract. The internal state labelling problem for the d-row irreducible representations of SU(n) (where $2d \leq n$), when reduced with respect to SO(n), is shown to amount to the external state labelling problem for U(d). The canonical solution of the latter due to Biedenharn *et al* provides a canonical solution of the former, which reflects the operation of Littlewood's branching rule for U(n) ⊃ O(n) in a very simple way.

1. Introduction

The usefulness of the group chain SU(n) ⊃ SO(n) in applications of group theory to both atomic and nuclear spectroscopy has been appreciated for a long time (Racah 1949, Elliott 1958, Moshinsky 1967, Wybourne 1970). Unfortunately it turns out that the SO(n) subgroup does not provide quantum numbers enough to specify completely the states transforming under an irreducible representation (irrep) of SU(n): this is the so-called state labelling problem for SU(n) ⊃ SO(n).

The prototype of such a labelling problem is that arising for the chain SU(3) ⊃ SO(3). Since the pioneering work of Elliott (1958), Bargmann and Moshinsky (1961) and Racah (1964), several different methods have been proposed to resolve this difficulty, and they can be divided into two classes (see e.g. Moshinsky *et al* 1975). The first type of solution uses as bases the common eigenstates of a complete set of commuting Hermitian operators. Besides the Casimir operators of group and subgroup, the latter contains an additional missing label operator. Its eigenvalues provide the missing label for the state vectors. They are not integer numbers but the corresponding basis states are orthonormal. The other type of solution leads to an integer additional label associated with analytic, but non-orthogonal, basis states. The latter are not the eigenstates of any complete set of commuting operators.

Among the second of the above approaches, the method of elementary permissible diagrams (Moshinsky and Syamala Devi 1969)—or the equivalent method of elementary multiplets (Sharp and Lam 1969)—plays an important part because it is based upon Littlewood's theorem (1950) for the reduction of an irrep of U(3) into irreducible parts with respect to O(3). The basis states corresponding to given irreps of SU(3) and SO(3), and of highest weight with respect to the latter group, are factorised into

† Maître de recherches FNRS.

a product of powers of a finite number of similar basis states associated with some low-dimensional irreps of SU(3) and SO(3) (characterised by the so-called elementary permissible diagrams). The power of one of these elementary basis states provides the missing label.

The generalisation of this method to other group chains, although clear in principle, is rather hard to perform because the number of elementary permissible diagrams tremendously increases with the rank of the Lie algebras involved. In the case of the $SU(n) \supset SO(n)$ chain, the application of this method has been restricted, as far as we know, to irreps of $U(n)$ which have no more than three rows (Syamala Devi 1970).

The purpose of the present series of papers is to propose a new solution to the state labelling problem for $SU(n) \supset SO(n)$, which reflects the operation of Littlewood's branching rule in a much simpler way than the method of elementary permissible diagrams while remaining free of such practical limitations. The present paper is concerned with the general formulation of the method. For this purpose, we shall restrict ourselves to the case where the number of rows d of the Young diagram characterising the $SU(n)$ irrep does not exceed $\nu = [n/2]$, i.e. the largest integer contained in $n/2$. When this condition is fulfilled, Littlewood's branching rule directly applies without need to supplement it with Newell's modification rules (1951). The way the latter can be included in the present picture for $SU(n)$ irreps for which $d > [n/2]$ will be dealt with in a forthcoming paper.

In § 2, we establish the equivalence between the state labelling problems for the complementary chains $U(n) \supset O(n)$ and $Sp(2d, R) \supset U(d)$ when d-row irreps of $U(n)$ are considered. In § 3, we construct the highest weight states of equivalent $O(n)$ irreps in a Bargmann space of analytic functions in dn complex variables. In § 4, we determine from them the highest weight states of equivalent $O(n)$ irreps belonging to a given $U(n)$ irrep, thereby solving the state labelling problem for $U(n) \supset O(n)$, or $SU(n) \supset SO(n)$. Section 5 contains some concluding remarks.

2. The state labelling problems for $U(n) \supset O(n)$ and $Sp(2d, R) \supset U(d)$

In the present paper, we shall discuss the state labelling problem for the d-row irreps of $SU(n)$ (where $2d \leq n$), when reduced with respect to $SO(n)$, in terms of the chain

$$U(n) \supset O(n). \tag{2.1}$$

As is well known (Wybourne 1970), the irreps $[h_1 h_2 \ldots h_d]$ of $U(n)$ remain irreducible under $SU(n)$ and are characterised by the same partition; the same is true for the irreps $(\lambda_1 \lambda_2 \ldots \lambda_d)$ of $O(n)$ when reduced with respect to $SO(n)$, except when $n = 2\nu$, $d = \nu$, and $\lambda_\nu > 0$, in which case

$$(\lambda_1 \lambda_2 \ldots \lambda_{\nu-1} \lambda_\nu) \rightarrow (\lambda_1 \lambda_2 \ldots \lambda_{\nu-1} \lambda_\nu) + (\lambda_1 \lambda_2 \ldots \lambda_{\nu-1} - \lambda_\nu). \tag{2.2}$$

Littlewood's branching theorem (1950) for chain (2.1) states that if in the reduction to irreps of $U(n)$, the product representation $[\lambda_1 \lambda_2 \ldots \lambda_d] \times [h_1^s h_2^s \ldots h_d^s]$ contains $[h_1 h_2 \ldots h_d]$ a certain number of times, which we denote by $g_{[\lambda_1 \lambda_2 \ldots \lambda_d][h_1^s h_2^s \ldots h_d^s][h_1 h_2 \ldots h_d]}$, then the irrep $[h_1 h_2 \ldots h_d]$ of $U(n)$ breaks into irreps $(\lambda_1 \lambda_2 \ldots \lambda_d)$ of $O(n)$ according to

$$[h_1 \ldots h_d] = \sum_{\lambda_1 \ldots \lambda_d} \left(\sum_{h_1^s \ldots h_d^s} g_{[\lambda_1 \ldots \lambda_d][h_1^s \ldots h_d^s][h_1 \ldots h_d]} \right) (\lambda_1 \ldots \lambda_d), \tag{2.3}$$

the summation in brackets being over all partitions that characterise the irreps of $\mathrm{U}(n)$ of the type $[h_1^s h_2^s \ldots h_d^s]$ with $h_1^s, \ldots, h_d^s$ even integers. Littlewood's theorem may be used for any d value such that $d \leqslant n$. However, when $2d > n$, $(\lambda_1 \ldots \lambda_d)$ is a non-standard symbol of $\mathrm{O}(n)$ and has to be converted into a standard one by using Newell's modification rules (1951). This difficulty will not arise in the present paper since we restrict ourselves to d values such that $2d \leqslant n$.

Let us consider the highest weight state (HWS) P of one of the equivalent $\mathrm{O}(n)$ irreps characterised by $(\lambda_1 \ldots \lambda_d)$ and contained in the irrep $[h_1 \ldots h_d]$ of $\mathrm{U}(n)$. This HWS can be built (Moshinsky 1963) as a polynomial in dn boson creation operators η_{is} (where $i = 1, 2, \ldots, d$, and $s = 1, 2, \ldots, n$). We shall use Bargmann's representation (1961), wherein the creation operators η_{is} are represented by some complex variables z_{is}, and the corresponding annihilation operators ξ_{is} by $\partial/\partial z_{is}$. In such a representation, the generators of the $\mathrm{U}(n)$ and $\mathrm{O}(n)$ groups are given by

$$C_{st} = \sum_{i=1}^{d} z_{is}\partial/\partial z_{it} \tag{2.4}$$

and

$$\Lambda_{st} = -\mathrm{i}(C_{st} - C_{ts}), \tag{2.5}$$

respectively.

The construction of the HWS $P(z_{is})$ is most easily discussed in terms of the group chain

$$\mathrm{Sp}(2d, R) \supset \mathrm{U}(d), \tag{2.6}$$

where $\mathrm{U}(d)$ is generated by the operators

$$C_{ij} = \sum_{s=1}^{n} z_{is}\partial/\partial z_{js}, \tag{2.7}$$

and $\mathrm{Sp}(2d, R)$ by

$$D_{ij}^{\dagger} = \sum_{s=1}^{n} z_{is}z_{js}, \qquad D_{ij} = \sum_{s=1}^{n} \frac{\partial^2}{\partial z_{is}\,\partial z_{js}}, \qquad E_{ij} = C_{ij} + \tfrac{1}{2}n\delta_{ij}. \tag{2.8}$$

The groups of chain (2.6) are complementary (Moshinsky and Quesne 1970) with respect to those of chain (2.1) within the irrep $\langle \frac{1}{2}^{dn} \rangle$ or $\langle \frac{1}{2}^{dn-1}\, \frac{3}{2} \rangle$ of a larger group $\mathrm{Sp}(2dn, R)$ (Moshinsky and Quesne 1971). In other words, all the boson states belonging to an irrep of $\mathrm{U}(d)$ $[\mathrm{O}(n)]$ characterised by $[h_1 \ldots h_d][(\lambda_1 \ldots \lambda_d)]$, and of highest weight with respect to this group, belong to a single irrep of $\mathrm{U}(n)$ $[\mathrm{Sp}(2d, R)]$ specified by $[h_1 \ldots h_d][\langle \lambda_d + n/2, \ldots, \lambda_1 + n/2 \rangle]$ (Moshinsky 1963, Moshinsky and Quesne 1971).

Consequently, $P(z_{is})$ can be written as

$$P(z_{is}) = \left\langle z_{is} \left| \begin{matrix} \langle \lambda_d + n/2, \ldots, \lambda_1 + n/2 \rangle \\ (\Gamma^s)[h_1 \ldots h_d] \\ \max \end{matrix} \right. ; \begin{matrix} [h_1 \ldots h_d] \\ (\Gamma^s)(\lambda_1 \ldots \lambda_d) \\ \max \end{matrix} \right\rangle, \tag{2.9}$$

where the left-hand part of the ket characterises the irreps of chain (2.6) while the right-hand one specifies those of chain (2.1), and $P(z_{is})$ is of highest weight with respect to both subgroups $\mathrm{U}(d)$ and $\mathrm{O}(n)$. The index (Γ^s) denotes the whole set of $d(d-1)/2$ missing labels (Racah 1965) which distinguish between equivalent irreps of $\mathrm{U}(d)$

contained in a given irrep of Sp(2d, R). At this point it is important to realise that due to the complementarity relationship between chains (2.1) and (2.6), the set (Γ^s) also provides us with the $d(d-1)/2$ missing labels characterising the equivalent irreps of O(n) contained in a given d-row irrep of U(n). We shall now proceed to show how an explicit construction of $P(z_{is})$ leads to a canonical definition of (Γ^s).

For such a purpose, let us consider the weight and raising generators of U(d) and O(n). For the former group, they are the operators C_{ii} and C_{ij}, $i<j$, respectively. For the latter, following Wong's notation (1967), they are given by

$$\begin{aligned} &H_\alpha = \Lambda_{2\alpha,2\alpha-1}, \\ &A^\beta_\alpha = \tfrac{1}{2}(\Lambda_{2\alpha-1,2\beta}+\Lambda_{2\alpha,2\beta-1}+\mathrm{i}\Lambda_{2\alpha-1,2\beta-1}-\mathrm{i}\Lambda_{2\alpha,2\beta}), \qquad \beta<\alpha, \qquad (2.10)\\ &D^\beta_\alpha = \tfrac{1}{2}(-\Lambda_{2\alpha-1,2\beta}+\Lambda_{2\alpha,2\beta-1}-\mathrm{i}\Lambda_{2\alpha-1,2\beta-1}-\mathrm{i}\Lambda_{2\alpha,2\beta}), \qquad \beta<\alpha, \\ &E^\alpha_n = 2^{-1/2}(\Lambda_{n,2\alpha}+\mathrm{i}\Lambda_{n,2\alpha-1}) \qquad \text{(only when } n=2\nu+1\text{)}, \end{aligned}$$

where α and β run from 1 to $\nu=[n/2]$.

The polynomials $P(z_{is})$ of (2.9), corresponding to all possible sets (Γ^s), are solutions of the following system of first-order differential equations:

$$\begin{aligned} &H_\alpha P = \lambda_\alpha P, \qquad && A^\beta_\alpha P = 0, \beta<\alpha, \qquad && (2.11a,b)\\ &D^\beta_\alpha P = 0, \beta<\alpha, && E^\alpha_n P = 0 \text{ (only when } n=2\nu+1), && (2.11c,d)\\ &C_{ii}P = h_i P, && C_{ij}P = 0, i<j, && (2.12a,b) \end{aligned}$$

where $\lambda_{d+1}=\ldots=\lambda_\nu=0$. The usual procedure to solve such a system of equations (see e.g. Bargmann and Moshinsky 1961) consists in considering (2.12) first and (2.11) afterwards, i.e. in looking among all the polynomials $P(z_{is})$ belonging to a given U(n) irrep for those which also belong to a given O(n) irrep and are of highest weight with respect to O(n). In the present paper, we shall reverse this procedure and solve (2.11) first, then (2.12). This means that among all the polynomials $P(z_{is})$ belonging to a given Sp(2d, R) irrep we look for those which also belong to a given U(d) irrep and are of highest weight with respect to U(d). The method followed here would therefore be the usual procedure for the complementary chain (2.6).

3. Highest weight states of equivalent O(n) irreps

Before solving the system of equations (2.11), it is advantageous to simplify it by making some appropriate changes of variables. First we note that the dn variables z_{is} do not have a definite weight with respect to O(n). In analogy with the spherical coordinates in three dimensions, we are led to define the linear combinations

$$\begin{aligned} &a_{i\alpha} = 2^{-1/2}(z_{i,2\alpha-1}-\mathrm{i}z_{i,2\alpha}), \qquad b_{i\alpha} = 2^{-1/2}(z_{i,2\alpha-1}+\mathrm{i}z_{i,2\alpha}), \qquad (3.1)\\ &c_i = z_{in} \qquad \text{only when } n=2\nu+1, \end{aligned}$$

where i runs from 1 to d and α from 1 to ν. The new variables $a_{i\alpha}$, $b_{i\alpha}$ and c_i have a definite weight with respect to O(n), respectively equal to $(0,\ldots,0,1,0,\ldots,0)$, $(0,\ldots,0,-1,0,\ldots,0)$, and $(0,\ldots,0)$, where ±1 stands on the αth place.

Next we introduce the scalars with respect to O(n) which can be built from the z_{is} variables,

$$w_{ij} = \sum_{s=1}^{n} z_{is} z_{js} = w_{ji}, \qquad i, j = 1, \ldots, d. \tag{3.2}$$

Since the O(n) generators do not act on the $\frac{1}{2}d(d+1)$ functions w_{ij}, it is interesting to go to a new set of variables including the w_{ij}'s because the O(n) generators will contain no derivatives with respect to them. In this way, it is possible to reduce the number of variables in (2.11). Let us therefore eliminate $\frac{1}{2}d(d+1)$ variables of the $b_{i\alpha}$ type—being of low weight, they are likely to play no part in the HWS—and replace them by the w_{ij}'s. The new variables are defined by

$$\begin{aligned}
u_{i\alpha} &= a_{i\alpha}, & \alpha &= 1, \ldots, \nu, \\
v_{i\rho} &= b_{i\rho}, & \rho &= 1, \ldots, \nu - i, \\
&= c_i, & \rho &= \nu - i + 1 \qquad \text{(only when } n = 2\nu + 1), \\
w_{ij} &= \sum_{\alpha=1}^{\nu} (a_{i\alpha} b_{j\alpha} + a_{j\alpha} b_{i\alpha}), & & \text{when } n = 2\nu, \\
&= \sum_{\alpha=1}^{\nu} (a_{i\alpha} b_{j\alpha} + a_{j\alpha} b_{i\alpha}) + c_i c_j, & & \text{when } n = 2\nu + 1,
\end{aligned} \tag{3.3}$$

where i and j run from 1 to d. It can be easily checked that the Jacobian of the transformation is different from zero so that the variables $u_{i\alpha}$, $v_{i\rho}$, and w_{ij} are functionally independent.

In terms of the variables $u_{i\alpha}$, $v_{i\rho}$, and w_{ij}, the weight and raising generators of O(n) become

$$\begin{aligned}
H_\alpha &= \sum_{i=1}^{d} u_{i\alpha} \frac{\partial}{\partial u_{i\alpha}} - \sum_{i=1}^{\min(\nu-\alpha, d)} v_{i\alpha} \frac{\partial}{\partial v_{i\alpha}}, \\
A_\alpha^\beta &= \sum_{i=1}^{\min(\nu-\beta, d)} u_{i\alpha} \frac{\partial}{\partial v_{i\beta}} - \sum_{i=1}^{\min(\nu-\alpha, d)} u_{i\beta} \frac{\partial}{\partial v_{i\alpha}}, \qquad \beta < \alpha, \\
D_\alpha^\beta &= \sum_{i=1}^{d} u_{i\beta} \frac{\partial}{\partial u_{i\alpha}} - \sum_{i=1}^{\min(\nu-\beta, d)} b_{i\alpha} \frac{\partial}{\partial v_{i\beta}}, \qquad \beta < \alpha, \\
E_n^\alpha &= \sum_{i=1}^{\min(\nu-\alpha, d)} v_{i,\nu-i+1} \frac{\partial}{\partial v_{i\alpha}} - \sum_{i=1}^{d} u_{i\alpha} \frac{\partial}{\partial v_{i,\nu-i+1}} \qquad \text{(only when } n = 2\nu + 1),
\end{aligned} \tag{3.4}$$

where $b_{i\alpha}$ is a function of the new variables, that could be obtained in principle by inverting (3.3).

Let us now look for the solutions $P(u_{i\alpha}, v_{i\rho}, w_{ij})$ of (2.11). Since the change of variables (3.3) is not linear, the functions $P(u_{i\alpha}, v_{i\rho}, w_{ij})$ could be non-analytic in $u_{i\alpha}$, $v_{i\rho}$ and w_{ij}, while remaining polynomials in z_{is}. We shall disregard here this possibility because, as shown in the next section, the analytic solutions provide us with the HWS of all the equivalent O(n) irreps contained in a given U(n) irrep in accordance with Littlewood's rule.

Equations (2.11b) and (2.11d) (the latter for $n = 2\nu + 1$) only contain derivatives with respect to the $v_{i\rho}$ variables. As shown in the appendix, they impose that P does not depend upon the latter. We are thus left with functions $P(u_{i\alpha}, w_{ij})$.

Taking (3.4) into account, the remaining equations (2.11a,c) become

$$H_\alpha P(u_{i\alpha}, w_{ij}) = \sum_{i=1}^{d} u_{i\alpha}\frac{\partial}{\partial u_{i\alpha}} P(u_{i\alpha}, w_{ij}) = \lambda_\alpha P(u_{i\alpha}, w_{ij}), \tag{3.5}$$

and

$$D_\alpha^\beta P(u_{i\alpha}, w_{ij}) = \sum_{i=1}^{d} u_{i\beta}\frac{\partial}{\partial u_{i\alpha}} P(u_{i\alpha}, w_{ij}) = 0, \qquad \beta < \alpha.$$

When acting upon functions $P(u_{i\alpha}, w_{ij})$, the operators H_α and D_α^β therefore respectively behave as the weight and raising generators of a U(ν) group, whose generator general form is $\Sigma_{i=1}^{d} u_{i\alpha}\, \partial/\partial u_{i\beta}$, α, $\beta = 1, \ldots, \nu$. Equation (3.5) means that $P(u_{i\alpha}, w_{ij})$ is the HWS of an irrep $[\lambda_1\lambda_2 \ldots \lambda_d]$ of this U(ν) group.

From the results of Moshinsky (1963), it is easy to construct particular solutions of (3.5), which only depend upon the $u_{i\alpha}$ variables. They are given by

$$\bar{P}(u_{i\alpha}) = (u_{11})^{\lambda_1-\lambda_2}(u_{12,12})^{\lambda_2-\lambda_3} \ldots (u_{12\ldots d-1,12\ldots d-1})^{\lambda_{d-1}-\lambda_d}$$
$$\times (u_{12\ldots d,12\ldots d})^{\lambda_d} Z\left(\frac{u_{i1}}{u_{11}}, \frac{u_{1i,12}}{u_{12,12}}, \ldots, \frac{u_{12\ldots d-2d,12\ldots d-1}}{u_{12\ldots d-1,12\ldots d-1}}\right). \tag{3.6}$$

In (3.6), $u_{12\ldots i-1j,12\ldots i-1i}$, $1 \leqslant i \leqslant j \leqslant d$, is defined by

$$u_{12\ldots i-1j,12\ldots i-1i} = \sum_p (-)^p u_{1,p(1)} u_{2,p(2)} \ldots u_{i-1,p(i-1)} u_{j,p(i)}, \tag{3.7}$$

where the summation is carried out over the $i!$ permutations of the indices $1, 2, \ldots, i-1, i$, and Z is an arbitrary polynomial in the variables indicated, subject to the condition that when multiplied by the other factors in (3.6), it should still be a polynomial in the $u_{i\alpha}$'s. The number of linearly independent functions $\bar{P}(u_{i\alpha})$ is equal to the dimension of the irrep $[\lambda_1 \ldots \lambda_d]$ of U(d). The general analytic solution of (2.11) is obtained by linearly combining these functions $\bar{P}(u_{i\alpha})$ with coefficients $P^s(w_{ij})$ which are arbitrary analytic functions of the w_{ij} variables.

4. Highest weight states of equivalent O(n) irreps belonging to a given U(n) irrep

To obtain the highest weight states of both O(n) and U(d) irreps, it remains to impose the conditions (2.12) on the linear combination of functions $\bar{P}(u_{i\alpha})P^s(w_{ij})$. When acting upon such functions, the U(d) generators (2.7) reduce to the expression

$$C_{ij} = \sum_{\alpha=1}^{\nu} u_{i\alpha}\frac{\partial}{\partial u_{j\alpha}} + \sum_{k=1}^{d} (1+\delta_{kj}) w_{ik}\frac{\partial}{\partial w_{jk}}, \tag{4.1}$$

where we have carried out the changes of variables (3.1) and (3.3). On the right-hand side, the first term operates on $\bar{P}(u_{i\alpha})$, whereas the second one acts on $P^s(w_{ij})$.

As mentioned above, the polynomials $\bar{P}(u_{i\alpha})$ corresponding to all possible independent choices for Z span the representation space of an irrep $[\lambda_1 \ldots \lambda_d]$ of U(d). It is possible to choose Z in such a way that the polynomials $\bar{P}(u_{i\alpha})$ transform irreducibly under the canonical chain $\mathrm{U}(d) \supset \mathrm{U}(d-1) \supset \ldots \supset \mathrm{U}(1)$, and are characterised by a Gel'fand pattern (λ) (Gel'fand and Tseitlin 1950). Then they can be written in analogy with (2.9) as

$$\bar{P}(u_{i\alpha})=\left\langle u_{i\alpha}\left|\begin{matrix}\langle\lambda_d+n/2,\ldots,\lambda_1+n/2\rangle\\ [\lambda_1\ldots\lambda_d]\\ (\lambda)\end{matrix}\;;\;\begin{matrix}[\lambda_1\ldots\lambda_d]\\ (\lambda_1\ldots\lambda_d)\\ \max\end{matrix}\right.\right\rangle, \tag{4.2}$$

where no additional label of the (Γ^s) type is needed.

Since the functions $P^s(w_{ij})$ depend upon scalar variables under O(n), they transform according to the irrep (0) of O(n), and span the representation space of an irrep $\langle n/2\ldots n/2\rangle$ of Sp($2d, R$). In a recent work (Deenen and Quesne 1982), we showed that the latter breaks into a direct sum of irreps $[h_1^s\ldots h_d^s]$ of U(d), where $h_1^s,\ldots,h_d^s$ are even and each irrep has a multiplicity one. We can therefore specify the independent analytic functions $P^s(w_{ij})$ by the partition $[h_1^s\ldots h_d^s]$ and the Gel'fand pattern (h^s) so that

$$P^s(w_{ij})=\left\langle w_{ij}\left|\begin{matrix}\langle n/2\ldots n/2\rangle\\ [h_1^s\ldots h_d^s]\\ (h^s)\end{matrix}\;;\;\begin{matrix}[h_1^s\ldots h_d^s]\\ (0)\\ \max\end{matrix}\right.\right\rangle. \tag{4.3}$$

The HWS of the irrep $[h_1^s\ldots h_d^s]$ is given by (Deenen and Quesne 1982)

$$\left\langle w_{ij}\left|\begin{matrix}\langle n/2\ldots n/2\rangle\\ [h_1^s\ldots h_d^s]\\ \max\end{matrix}\;;\;\begin{matrix}[h_1^s\ldots h_d^s]\\ (0)\\ \max\end{matrix}\right.\right\rangle=A_{h_1^s\ldots h_d^s}\prod_{i=1}^{d}(w_{12\ldots i,12\ldots i})^{(h_i^s-h_{i+1}^s)/2}, \tag{4.4}$$

where $w_{12\ldots i,12\ldots i}$ is defined in terms of w_{kl} by an expression similar to (3.7), $A_{h_1^s\ldots h_d^s}$ is some normalisation coefficient, and $h_{d+1}^s=0$. Therefore, by applying appropriate U(d) lowering operators (Nagel and Moshinsky 1965) to this HWS, we can actually construct the right-hand side of (4.3).

From (4.2) and (4.3), the solutions of (2.12a, b) are now obtained in a straightforward way: we only have to couple $\bar{P}(u_{i\alpha})$ and $P^s(w_{ij})$ to a definite irrep $[h_1\ldots h_d]$ of U(d) by means of appropriate U(d) Wigner coefficients. The result can be written as

$$\begin{aligned}&\left\langle u_{i\alpha}, w_{ij}\left|\begin{matrix}\langle\lambda_d+n/2,\ldots,\lambda_1+n/2\rangle\\ (\Gamma^s)[h_1\ldots h_d]\\ \max\end{matrix}\;;\;\begin{matrix}[h_1\ldots h_d]\\ (\Gamma^s)(\lambda_1\ldots\lambda_d)\\ \max\end{matrix}\right.\right\rangle\\ &\quad=\sum_{(h^s)(\lambda)}\left\langle\begin{matrix}[h_1\ldots h_d]\\ \max\end{matrix}\right|\left\langle\begin{matrix}(\gamma^s)\\ [h_1^s\ldots h_d^s]\\ (h^s)\end{matrix}\right\rangle\left|\begin{matrix}[\lambda_1\ldots\lambda_d]\\ (\lambda)\end{matrix}\right\rangle\\ &\qquad\times\left\langle u_{i\alpha}\left|\begin{matrix}\langle\lambda_d+n/2,\ldots,\lambda_1+n/2\rangle\\ [\lambda_1\ldots\lambda_d]\\ (\lambda)\end{matrix}\;;\;\begin{matrix}[\lambda_1\ldots\lambda_d]\\ (\lambda_1\ldots\lambda_d)\\ \max\end{matrix}\right.\right\rangle\\ &\qquad\times\left\langle w_{ij}\left|\begin{matrix}\langle n/2\ldots n/2\rangle\\ [h_1^s\ldots h_d^s]\\ (h^s)\end{matrix}\;;\;\begin{matrix}[h_1^s\ldots h_d^s]\\ (0)\\ \max\end{matrix}\right.\right\rangle,\end{aligned} \tag{4.5}$$

where we use Biedenharn's notations for the U(d) Wigner coefficient (Biedenharn *et* *al* 1967). The operator pattern (γ^s) solves the state labelling problem for the product $[\lambda_1 \ldots \lambda_d] \times [h_1^s \ldots h_d^s]$ of U(d) irreps. Together with the partition $[h_1^s \ldots h_d^s]$, it can serve to distinguish between equivalent irreps of U(d) [O(n)] contained in a given irrep of Sp($2d, \boldsymbol{R}$) [U(n)]. In other words (Γ^s) can be taken as

$$(\Gamma^s) = \begin{pmatrix} (\gamma^s) \\ [h_1^s \ldots h_d^s] \end{pmatrix}. \tag{4.6}$$

This definition of (Γ^s) indeed provides us with the right number $d(d-1)/2$ of additional independent labels since the Γ_{ij}^s are linked by the relations

$$\sum_{j=1}^{i} \Gamma_{ji}^s - \sum_{j=1}^{i-1} \Gamma_{ji-1}^s = h_i - \lambda_i, \qquad i = 1, \ldots, d. \tag{4.7}$$

Moreover the states corresponding to different (Γ^s) patterns are linearly independent by construction.

We have therefore proved† that the internal state labelling problem for the d-row irreps of U(n) (where $2d \leq n$), when reduced with respect to O(n), amounts to the external state labelling problem for U(d), for which a canonical solution is known (Biedenharn *et al* 1967). This new solution of the state labelling problem for U(n) ⊃ O(n)—or for SU(n) ⊃ SO(n)—may also be termed canonical since it reflects the operation of Littlewood's branching rule in a straightforward way, as can be seen by comparing (2.3) and (4.5).

Since the set of labels (Γ^s) are integer, we should expect according to Racah (1964) that the corresponding bases are not orthogonal. Although two states (4.5) corresponding to the same irrep $[h_1^s \ldots h_d^s]$ but to different patterns (γ^s) are orthogonal, this is not true in general for two states associated with different irreps $[h_1^s \ldots h_d^s]$. We indeed note that in (4.5) both $\bar{P}(u_{i\alpha})$ and $P^s(w_{ij})$ are classified according to the same U(d) group. Had they belonged to given irreps of two commuting U(d) groups, orthogonality with respect to $[h_1^s \ldots h_d^s]$ would have resulted.

5. Conclusion

In the present paper, we have taken advantage of the complementarity relationship between the chains U(n) ⊃ O(n) and Sp($2d, \boldsymbol{R}$) ⊃ U(d) for d-row irreps of U(n) to propose a new solution of the state labelling problem for U(n) ⊃ O(n) or SU(n) ⊃ SO(n). This solution is not restricted to small values of n or d (provided that $2d \leq n$), although its practical usefulness is of course limited by the need for an explicit knowledge of the U(d) Wigner coefficients.

Since it is directly connected with Littlewood's branching rule for U(n) ⊃ O(n), it should be intimately related to the method of elementary permissible diagrams. In forthcoming papers, we plan to study this point as well as to extend the present analysis to irreps of U(n) for which $2d > n$.

The same kind of approach as that developed here could be used to solve the state labelling problem for the chain U(2ν) ⊃ Sp(2ν), for which both a Littlewood's branching rule (1943) and a chain of complementary groups (Quesne 1973) are known.

† In the case $d = 3$, a similar result was obtained by Vasilevskii *et al* (1980) using a different approach.

Appendix. Solutions of (2.11*b*) and (2.11*d*)

In the present appendix, we wish to show that the solutions $P(u_{i\alpha}, v_{i\rho}, w_{ij})$ of (2.11*b*) and (2.11*d*) are those functions which only depend upon the $u_{i\alpha}$ and w_{ij} variables.

Let us first consider the set of equations (2.11*b*). In the variables $u_{i\alpha}$, $v_{i\rho}$ and w_{ij}, they can be written as

$$A_\alpha^\beta P = \left(\sum_{i=1}^{\min(\nu-\beta,d)} u_{i\alpha} \frac{\partial}{\partial v_{i\beta}} - \sum_{i=1}^{\min(\nu-\alpha,d)} u_{i\beta} \frac{\partial}{\partial v_{i\alpha}} \right) P = 0, \qquad \beta < \alpha. \tag{A1}$$

For any β value between 1 and $\nu - 1$, there are $\nu - \beta$ equations corresponding to $\alpha = \beta + 1, \ldots, \nu$. Let us arrange the β values in decreasing order, starting from $\beta = \nu - 1$. For this value, there is a single equation

$$A_\nu^{\nu-1} P = u_{1\nu} \frac{\partial P}{\partial v_{1,\nu-1}} = 0, \tag{A2}$$

which imposes that P does not depend upon $v_{1,\nu-1}$. Let us show by induction over β that the equations corresponding to $\beta = \nu - 1, \nu - 2, \ldots, \beta_0$ impose that P does not depend upon $v_{i\beta}$, $i = 1, \ldots, \min(\nu - \beta, d)$, $\beta = \nu - 1, \ldots, \beta_0$. If the latter is true for $\beta = \nu - 1, \nu - 2, \ldots, \beta_0 + 1$, when considering $\beta = \beta_0$, we have to solve $\nu - \beta_0$ additional linear equations

$$A_\alpha^{\beta_0} P = \sum_{i=1}^{\min(\nu-\beta_0,d)} u_{i\alpha} \frac{\partial P}{\partial v_{i\beta_0}} = 0, \qquad \alpha = \beta_0 + 1, \ldots, \nu, \tag{A3}$$

in $\nu - \beta_0$ or d unknowns $\partial P / \partial v_{i\beta_0}$ according to which one is the smallest. Since the matrix of the coefficients coincides with that of the $u_{i\alpha}$ variables for which $i = 1, \ldots, \min(\nu - \beta_0, d)$, and $\alpha = \beta_0 + 1, \ldots, \nu$, its rank is equal to the number of unknowns and the latter are therefore equal to zero. This completes the proof of the proposition. It then follows that the whole set of equations (2.11*b*) imposes that P does not depend upon $v_{i\rho}$, where $i = 1, \ldots, \min(\nu - \rho, d)$, and $\rho = 1, \ldots, \nu - 1$.

In the case where $n = 2\nu$, we have therefore succeeded in eliminating all the $v_{i\rho}$ variables. In the cases where $n = 2\nu + 1$, we are left with the variables $v_{i,\rho-i+1}$, $i = 1, \ldots, d$, but we have still to consider the set of equations (2.11*d*), which can be written as

$$E_n^\alpha P = \sum_{i=1}^{d} u_{i\alpha} \frac{\partial P}{\partial v_{i,\nu-i+1}} = 0, \qquad \alpha = 1, \ldots, \nu. \tag{A4}$$

This is a system of ν linear equations in the d $(\leq \nu)$ unknowns $\partial P / \partial v_{i,\nu-i+1}$. Since the matrix of the coefficients is of rank d, the unknowns are equal to zero, so that P does not depend upon $v_{i,\nu-i+1}$, $i = 1, \ldots, d$, either.

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